

THE UNIVERSITY OF AKRON
Mathematics and Computer Science

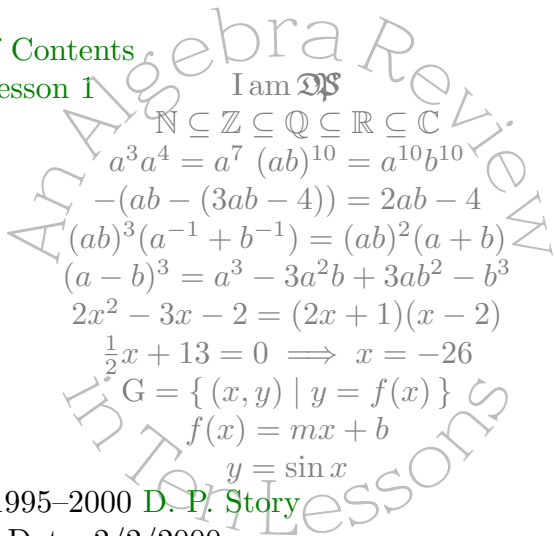


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menu

Lesson 1: Setting Up the Environment

Directory

- [Table of Contents](#)
- [Begin Lesson 1](#)



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Lesson 1: Setting Up the Environment

Table of Contents

1. Setting Up the Environment

1.1. The Real Number System, and friends

- The Natural Numbers • The Integers • The Rational Numbers • The Irrational Numbers • The Real Numbers

1.2. The Number Line, and relations

- Less than and Greater than

1.3. Distance and Absolute Value

- Absolute Value • Distance between two numbers • The Midpoint between two Numbers

1.4. Interval Notation

1. Setting Up the Environment

In this lesson, we review some very basic ideas and terminology of the so-called **Real Number System**. Do not take this first lesson lightly, your knowledge of the real number system and its properties is key to your understanding why certain **algebraic** manipulations are permissible, and why others are not. (When you are manipulating algebraic quantities, you are, in fact, manipulating numbers.)

Terminology is important in all scientific, technical and professional fields, and mathematics is no different, it has a lot of it. When you speak or write, you use words; the words you use must be understood by the ones with whom you are trying to communicate. For someone to understand your communication, the words must have a universal meaning; therefore, it is necessary for you to use the correct terminology to be able to effectively and efficiently communicate with others. Correct use of (mathematical) terminology is a sign that you understand what you are saying or writing.

1.1. The Real Number System, and friends

A pedestrian description of the real number system is that it is the set of all *decimal numbers*. Here are a few examples of decimal numbers:

1.456445 < Positive Number with Finite Decimal Expansion

-3.54 < Negative Number with Finite Decimal Expansion

1.49773487234... < Positive Number with Infinite Decimal Expansion

-3.4×10^9 < Negative Number in Scientific Notation

5.65445 E -5 < Positive Number in Scientific Notation

You have been working with these representations of numbers all your life. You have worked long and hard to master the mechanics of adding, subtracting, multiplying, and dividing these numbers. More recently, the **calculator** makes many of the operations just mentioned automatic and routine, but the calculator does not diminish the need to still have a ‘pencil and paper’ understanding of these operations, nor does it diminish the need to be able to read, interpret and convert numbers.

Decimal numbers are, in fact, the end product of a series of preliminary definitions. As important as decimal numbers are, let's leave them for now to discuss a series of definitions that lead ultimately to the construction of the real number system.

- **The Natural Numbers**

The *natural numbers* are the numbers

$$1, 2, 3, 4, 5, 6, \dots, 100, \dots, 1000, \dots$$

and so on *ad infinitum*. Mathematicians put braces around these numbers to create the *set of all natural numbers*:

$$\mathbb{N} = \{ 1, 2, 3, 4, 5, 6, \dots, n, \dots \}. \tag{1}$$

The letter \mathbb{N} is typically used to denote this set; i.e., \mathbb{N} symbolically represents the set of all natural numbers.

- **The Integers**

The *integers* consist of the natural numbers, the negative natural numbers and 0; or more succinctly:

$$\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots$$

The set of all integers is denoted by \mathbb{Z} (for some reason).

$$\mathbb{Z} = \{\dots, -4, -3, -2, -1, 0, 1, 2, 3, 4, \dots\}. \quad (2)$$

The set of integers \mathbb{Z} is an infinite set as well.

Question. Which of the following is true?

- (a) $\mathbb{N} \subseteq \mathbb{Z}$ (b) $\mathbb{Z} \subseteq \mathbb{N}$

Question. If p is an integer, what can be said about $2p$?

- (a) $2p$ is an *odd number* (b) $2p$ is an *even number*

An integer is either *even* or *odd*. Even and odd integers are often dealt with in mathematics. It is important to have a general representation of these types of numbers. In the previous **Question**, we observed that if $p \in \mathbb{Z}$, then $2p$ is an even integer. An odd integer can be thought

as any integer that follows an even integer. Since, in general, $2p$ is an even integer, $2p + 1$ must be an odd integer. Thus,

$$\begin{aligned} 2p \text{ is even, for any } p \in \mathbb{Z} \\ 2p + 1 \text{ is odd, for any } p \in \mathbb{Z} \end{aligned} \tag{3}$$

Side Bar. Do you know what the symbol $p \in \mathbb{Z}$ means? Yes or **No**.

Question. Is the product of two odd integers, an odd integer or an even integer?

- (a) Odd (b) Even (c) Can't tell

Mathematics has a lot of little facts like the one above. You too can discover interesting little tidbits—all you need is an inquiring mind.

• The Rational Numbers

The next step in the process of constructing the real number system is to create the *rational number system*. A rational number is a number

Section 1: Setting Up the Environment

of the form p/q , where $p, q \in \mathbb{Z}$ and $q \neq 0$. The letter used to denote the set of all rational numbers is \mathbb{Q} :

$$\mathbb{Q} = \left\{ \frac{p}{q} \mid p, q \in \mathbb{Z}, q \neq 0 \right\} \quad (4)$$

You have been dealing with rational numbers for most of your rational life. Here are a few examples,

$$\frac{1}{2}, \frac{12}{23}, -\frac{3}{4}, \frac{12345}{54321}, -\frac{8}{12}. \quad (5)$$

• **Question.** What is ‘wrong’ with the last number in list (5)? Think about your answer then click on the green dot (it’s a bullet actually).

Rational numbers are important in the world of computers because these are the only kind of numbers a computer knows! To your calculator, all numbers are rational.

Quiz. Here are some statements about rational numbers. Think before you respond. A passing grade is 100%.

Section 1: Setting Up the Environment

1. What is the relationship between \mathbb{Q} and \mathbb{Z} ?
(a) $\mathbb{Q} \subseteq \mathbb{Z}$ (b) $\mathbb{Z} \subseteq \mathbb{Q}$ (c) n.o.t.
2. Is the product of two rational numbers again a rational number?
(a) False (b) True
3. Is the sum of two rational numbers again a rational number?
(a) False (b) True

End Quiz.

Rationals and Decimals. Each rational number has a decimal representation.

1. Some rational numbers have a *finite* decimal expansion:

$$\frac{3}{4} = 0.75 \qquad -\frac{46}{125} = -0.368 \qquad \frac{27}{2} = 13.5$$

2. Some rational numbers have an *infinite repeating* decimal expansion:

$$\frac{2}{3} = 0.66666666\dots \qquad \frac{123}{999} = 0.123123123123\dots$$

3. The decimal expansion of some rational numbers is not unique; i.e., some rational numbers have *two* decimal expansions. For example,

$$\frac{3}{4} = 0.75 = 0.7499999999999999 \dots$$

Generally, any rational number that has a *finite decimal expansion* also has an *infinite repeating* decimal expansion:

$$32341.424552 = 32341.4245519999999999 \dots$$

$$0.5 = 0.4999999999999999 \dots$$

$$-12.43 = -12.4299999999999999 \dots$$

Do you get the idea?

EXERCISE 1.1. Write the decimal number 423.43537 as an *infinite repeating decimal*.

Yeah, but what about this exercise? It may be a tricky one!

EXERCISE 1.2. Write the decimal number 56.0 as an *infinite repeating decimal*.

Any decimal number with an infinite repeating decimal expansion is a rational number. Do you know how to compute this number? (Yes or **No**) There is a nice little trick.

EXAMPLE 1.1. Consider the decimal number:

$$453.123123123123123\dots$$

Convert this number to a rational number.

Now try one yourself.

EXERCISE 1.3. Convert the infinite repeating decimal number

$$3.45454545454545\dots$$

to a rational number.

Here's a slightly trickier one.

EXERCISE 1.4. Convert the decimal number

$$4.177777777777777 \dots$$

to a rational number.

- **The Irrational Numbers**

As the old phrase goes, “All other numbers that are *not rational* are called *irrational numbers*.” But are there “other numbers”? Yes, of course. The numbers $\sqrt{2}$, $\sqrt{3}$, and π are irrational numbers.

Characteristic of irrational numbers is their decimal expansions are *infinite* and *nonrepeating*. One famous example is π ; its decimal expansion, or at least the first 30 digits, is

$$\pi = 3.14159265358979323846264338328 \dots$$

- **The Real Numbers**

Section 1: Setting Up the Environment

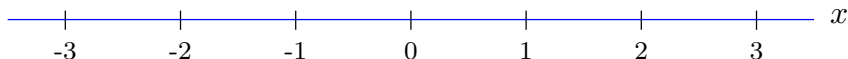
The set of all *real numbers*, denoted by \mathbb{R} , consists of all the **rational numbers** and all the **irrational numbers**. In symbols,

$$\mathbb{R} = \{ x \mid x \text{ is a rational or an irrational number} \} \quad (6)$$

In the sections that follow, some of the basic properties of the real number system are brought out. These properties form the basis for the construction of an *algebraic system*.

1.2. The Number Line, and relations

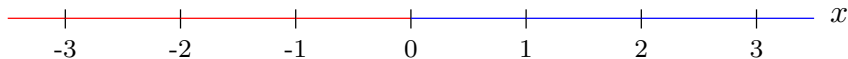
The real number system, \mathbb{R} , can be visualized by an horizontal line, typically called a *number line*. Usually, the number line is labeled using a symbol such as x , y , or z ; in this case the number line is called the x -axis, the y -axis, or the z -axis, respectively. We'll label our number line the x -axis.



The x -axis

Numbers to the *right* of zero (0) are called *positive numbers*. Numbers to the *left* of zero (0) are called *negative numbers*.

Section 1: Setting Up the Environment



The x -axis

The numbers in **blue** are the positive real numbers and the numbers in **red** are the negative real numbers.

• **Less than and Greater than**

Let a and b be distinct real numbers. (Distinct means $a \neq b$.) We say that a is *less than* b provided $a - b$ is a negative number; in this case we write $a < b$:

$$a < b \text{ means } a - b \text{ is negative.}$$

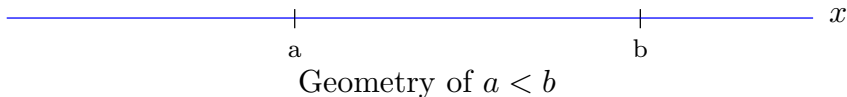
We say that a is *greater than* b provided $a - b$ is a positive number; in this case we write $a > b$:

$$a > b \text{ means } a - b \text{ is positive.}$$

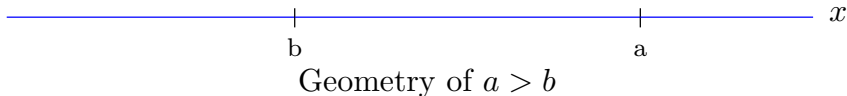
Variations on these definitions are $a \leq b$ (a is less than or equal to b) and $a \geq b$ (a is greater than or equal to b).

Section 1: Setting Up the Environment

Geometrically, what do these relations mean? On the x -axis, $a < b$ is interpreted as the number a lies to the *left* of b .



Similarly, the geometry of $a > b$ is that a lies to the right of b .



Of course, if a lies to the left of b ($a < b$) then b must lie to the right of a ($b > a$)! Understand?

Here are a couple of questions to test your knowledge of inequalities. Read the questions carefully, draw an x -axis to illustrate the situation, and then respond to the question. Don't guess. A passing score is 100%.

Question. If the number c lies to the right of the number d , then

(a) $c < d$

(b) $d < c$

(c) $c \leq d$

(d) $d \leq c$

Section 1: Setting Up the Environment

Question. If the number c *does not* lie to the left of the number d then

- (a) $c < d$ (b) $d < c$ (c) $c \leq d$ (d) $d \leq c$

Hopefully the use of different letters did not disturb you psychologically.

Double Sided Inequalities. A double inequality is an inequality of the form $a \leq x \leq b$, where a , b , and x are numbers. The meaning of this symbolism is

$$a \leq x \leq b \text{ means } a \leq x \text{ and } x \leq b,$$

which can be rephrased as

$$a \leq x \leq b \text{ means } x \text{ is between } a \text{ and } b.$$

EXERCISE 1.5. Critique the following double inequality: $1 \leq x \leq -2$.

1.3. Distance and Absolute Value

In mathematics, we often want to compute the distance one number is away from another. As a first case, let's compute the distance any given number is away from zero (0); this is a natural way to introduce the *absolute value* of a number.

- **Absolute Value**

How do you calculate the distance one number is away from zero? Begin by example. The number 5 is '5 units' away from 0. The number 54.233 is '54.233 units' away from 0. That was simple. Now for negative numbers. The number -10 is '10 units' from 0 and the number -43.6 is '43.6 units' from 0.

There is a pattern here. If x is a positive number, then x is ' x units' away from 0. Now if x is a negative number, the distance from 0 is obtained by 'dropping' the negative sign on the number x . The way we can do this is by prefixing the number x with a negative sign. That negative sign will cancel with the negative sign built into the number x . (For example, the negative sign in -5 can be removed by prefixing a

Section 1: Setting Up the Environment

negative sign: $-(-5) = 5$.) Returning to generality, a negative number x is ‘ $-x$ units’ away from 0.

Summary. Summarizing the previous paragraph we have

$$\text{The distance between } x \text{ and } 0 \text{ is } = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases}$$

The case study on the right-hand of the above equation is an important and deserves its own definition.

Absolute Value. Let x be a number, the *absolute value* of x , denoted by $|x|$, and is defined by

$$|x| = \begin{cases} x & \text{if } x \geq 0 \\ -x & \text{if } x < 0 \end{cases} \quad (7)$$

Section 1: Setting Up the Environment

This function is often a source of a lot of confusion for students. You must try to understand it on two levels: The Geometric interpretation of absolute value, and the technical definition (equation (7)).

When working with a numerical value, usually there is no problem with using the absolute value: Who could miscalculate

$$|4.3| = 4.3 \text{ or } |-6.24| = 6.24$$

incorrectly? Problems are encountered when we are working with *symbolic quantities*.

EXAMPLE 1.2. Suppose a is a number such that $a > 5$. Compute $|a - 5|$ and $|5 - a|$.

Then working with absolute values involving symbolic quantities, to remove the absolute values we must know the sign of this symbolic quantity. In **EXAMPLE 1.2**, we could remove the absolute value from $|a - 5|$ because we knew that $a - 5 > 0$.

Here's an easy one for you.

EXERCISE 1.6. Suppose $w < 4$. Compute $|w - 4|$.

A slight variation on the same theme is demonstrated in the next ...

EXAMPLE 1.3. Suppose s is a number such that $s > 6$. Compute $|2s - 3|$.

Now you investigate the following question.

EXERCISE 1.7. Suppose t is a number such that $t < -3$. Compute $|4t + 5|$.

The important point when dealing the absolute values: To simplify them, you must have enough information to deduce the sign to the quantity under consideration.

Sometimes, in mathematics, we are taking absolute values of symbolic quantities and *nothing is known about the sign of the quantity*. This occurs when we want to graph an expression or solve an equation involving absolute values. In this case, removal of the absolute value takes on a *conditional flavor*; to do this, we simply invoke the **definition** of absolute value.

EXAMPLE 1.4. Remove the absolute value in $|4x - 2|$.

This ‘case analysis’ is an important skill that you should have. Practice on the following two exercise. Solve and simplify completely before looking at the solutions.

EXERCISE 1.8. Remove the absolute value in $|3x - 12|$.

EXERCISE 1.9. Remove the absolute values in the expression $|2 + 3x|$.

Hopefully, you get the idea.

This kind of analysis can be carried out to various degrees of success involving arbitrarily complex expressions.

Regardless of the complexity of the expression, we can always remove the absolute value. For example, we can write $|x \sin(x^2)|$ as

$$|x \sin(x^2)| = \begin{cases} x \sin(x^2) & \text{if } x \sin(x^2) \geq 0 \\ -x \sin(x^2) & \text{if } x \sin(x^2) < 0 \end{cases}$$

Section 1: Setting Up the Environment

However, to continue the simplification, we basically need to solve the complex inequality $x \sin(x^2) \geq 0$. (This is not as hard as it might seem.)

The absolute value interacts with multiplication and division nicely. Here are some basic properties of absolute value.

Properties of Absolute Value

Let a and b be numbers, then

$$1. \quad |-a| = |a| \quad \text{and} \quad 2. \quad |ab| = |a||b| \quad \text{and} \quad \left| \frac{a}{b} \right| = \frac{|a|}{|b|}$$

Equation (1) simply says that $-a$ and a are the same distance away from zero.

Illustration 1. Here are some examples of these properties.

(a) $|5x| = |5||x| = 5|x|$.

(b) $|-7w| = |-7||w| = 7|w|$.

Section 1: Setting Up the Environment

(c) $|-z| = |z|$

(d) $|w^4| = |w| |w| |w| |w| = |w|^4.$

(e) $|a^n| = |a|^n$, as demonstrated in the previous example.

(f) $\left| \frac{-x}{9} \right| = \frac{|x|}{9}.$ ■

EXERCISE 1.10. Simplify each of the following.

(a) $|-56a|$ (b) $|(-1)^n x|$ (c) $\left| \frac{x}{-10} \right|$

When we deal with solving equations and inequalities, which is taken up in **Lesson 7** and **Lesson 8**, we'll enumerate other useful properties of the absolute value.

• Distance between two numbers

Given two numbers a and b , what is the distance between these numbers? Let's have a notation of

$$d(a, b) = \text{distance between } a \text{ and } b.$$

Section 1: Setting Up the Environment

For example, take $a = 5$ and $b = 9$. It is quite within your experience to agree with me that $d(5, 9) = 4$ ‘units’; that is, the number 5 and the number 9 are 4 units apart. We reasoned as follows: 9 is greater than 5 and so

$$d(5, 9) = \text{larger minus smaller} = 9 - 5 = 4.$$

What now is $d(-5, -12)$? Again, we first discern the larger of the two numbers. In this case, $-12 < -5$. Thus,

$$d(-5, -12) = \text{larger minus smaller} = -5 - (-12) = -5 + 12 = 7.$$

Thus, $d(-5, -12) = 7$.

As a general rule,

$$d(a, b) = \text{larger minus smaller of } a \text{ and } b.$$

But this is a rather silly formula; let’s get fancier to confuse you!

Section 1: Setting Up the Environment

Notice that if a and b are numbers, then from the **definition** of absolute value,

$$|a - b| = \begin{cases} a - b & \text{if } a - b \geq 0 \\ -(a - b) & \text{if } a - b < 0 \end{cases}$$

This can be rewritten as

$$|a - b| = \begin{cases} a - b & \text{if } a \geq b \\ b - a & \text{if } a < b \end{cases}$$

If you stare at this last equation long enough, you can see that

$$|a - b| = \text{larger minus smaller of } a \text{ and } b.$$

The distance between two numbers: Let a and b be numbers, then the distance between a and b is given by

$$d(a, b) = |a - b|. \tag{8}$$

This is a powerful and important interpretation of the absolute value.

EXERCISE 1.11. How far is 34 away from -5.3 ?

EXAMPLE 1.5. What can you say about the number x if a clue to its identity is $|x - 3| < 1$?

EXERCISE 1.12. What can be said about the number x if it is known that $|x - 10| < 3$?

EXERCISE 1.13. What can be said about the number x if it is known that $|x + 4| < 3$? (*Hint:* $|x + 4| = |x - (-4)|$.)

Here is the other type of inequality, see if you can interpret it.

EXERCISE 1.14. What can be said about the number x if it is known that $|x - 1| > 6$?

- **The Midpoint between two Numbers**

Finding the number exactly halfway between two others is a fundamental skill. Let $x_1 < x_2$ be numbers, and let \bar{x} be the midpoint

between them. Thus,

$$x_1 < \bar{x} < x_2$$

and,

$$d(x_1, \bar{x}) = \frac{1}{2}d(x_1, x_2) \quad (10)$$

The last equation reflects the facts that \bar{x} is halfway between x_1 and x_2 .

Now we use the distance formula, equation (8), to represent each of the above distances in (10).

$$\begin{aligned}d(x_1, \bar{x}) &= |x_1 - \bar{x}| \\ &= \bar{x} - x_1 \quad \triangleleft \text{since } x_1 < \bar{x} \text{ and (8)} \\ d(x_1, x_2) &= |x_1 - x_2| \\ &= x_2 - x_1 \quad \triangleleft \text{since } x_1 < x_2 \text{ and (8)}\end{aligned}$$

Section 1: Setting Up the Environment

Substitute these into equation (10) to obtain

$$\text{Given: } d(x_1, \bar{x}) = \frac{1}{2}d(x_1, x_2)$$

$$\text{Substitute: } \bar{x} - x_1 = \frac{1}{2}(x_2 - x_1)$$

$$\text{Solve for } \bar{x}: \quad \bar{x} = x_1 + \frac{1}{2}x_2 - \frac{1}{2}x_1$$

$$\bar{x} = \frac{1}{2}x_1 + \frac{1}{2}x_2$$

$$\bar{x} = \frac{x_1 + x_2}{2}$$

We have calculate the midpoint between two numbers:

Midpoint formula for the Real Line:

Let x_1 and x_2 be two numbers, then the number \bar{x} that is halfway between x_1 and x_2 is

$$\bar{x} = \frac{x_1 + x_2}{2} \tag{11}$$

Or, in other words, the midpoint between two numbers x_1 and x_2 is their (arithmetic) *average*.

Let's illustrate the calculations using a quiz format.

Quiz. Answer each of the following. Passing is 100%.

1. What number is midway between 2 and 9?
(a) 3.5 (b) 4.5 (c) 5.5 (d) n.o.t.
2. What number is midway between -12 and 34 ?
(a) 11 (b) 13 (c) 23 (d) n.o.t.
3. Is the point halfway between two rational numbers a rational number?
(a) Yes (b) No
4. Is the point midway between two irrational numbers an irrational number?
(a) Yes (b) No
5. Is the point midway between two integers an integer?
(a) Yes (b) No

EndQuiz

1.4. Interval Notation

In mathematics we deal with intervals of numbers. The most immediate application of intervals is that they are used to describe the solution sets to inequalities, but this will not happen in a disciplined way until **Lesson 8**.

In this section we introduce important notation for intervals.

Let's make a series of definitions. Let a and b be numbers with $a < b$.

$$(a, b) = \{x \mid a < x < b\} \quad \triangleleft \text{an open interval}$$

$$[a, b) = \{x \mid a \leq x < b\} \quad \triangleleft \text{closed on left, open on right}$$

$$(a, b] = \{x \mid a < x \leq b\} \quad \triangleleft \text{open on left, closed on right}$$

$$[a, b] = \{x \mid a \leq x \leq b\} \quad \triangleleft \text{a closed interval}$$

Section 1: Setting Up the Environment

These are intervals of *finite length*. We also have intervals of *infinite length*.

$$(a, \infty) = \{x \mid x > a\}$$

$$[a, \infty) = \{x \mid x \geq a\}$$

$$(-\infty, a) = \{x \mid x < a\}$$

$$(-\infty, a] = \{x \mid x \leq a\}$$

This point marks the end of this lesson. The previous section is rather sketchy at this time and will be revised and extended some time in the near future.

If you want to continue with **LESSON 2** you are most welcome to. The answer section follows so bail out now.

Solutions to Exercises

- 1.1.** Take the last digit decrease by one and post fix infinity many 9's.

$$423.43537 = 423.4353699999999999999999 \dots$$

Exercise 1.1. ■

1.2. Answer:

$$56.0 = 55.999999999999999999 \dots$$

Exercise 1.2. ■

1.3. Let $N = 3.45454545454545\dots$. Multiply this by 100 to obtain $100N = 345.45454545\dots$. Now subtract the two

$$\begin{aligned}99N &= 100N - N = 345.45454545454545\dots - 3.45454545454545\dots \\ &= 345 - 3 \\ &= 342\end{aligned}$$

Thus,

$$N = \frac{342}{99}.$$

which can be reduced to

$$N = \frac{38}{11}.$$

Or, $N = 3\frac{5}{11}$ if you prefer it written this way.

Exercise 1.3. ■

1.4. Let $N = 4.1777777777777777 \dots$

$$N = 4.1777777777777777 \dots$$

$$10N = 41.7777777777777777 \dots$$

$$100N = 417.7777777777777777 \dots$$

Subtract the second equation from the third to obtain

$$90N = 417 - 41 = 376$$

Thus,

$$N = \frac{376}{90}$$

But wait, time out, there is a common factor present in the above solution. A better answer is then,

$$N = \frac{188}{45}$$

1.5. The inequality is nonsense! From the interpretations just discussed, $1 \leq x \leq -2$ means $1 \leq x$ *and* $x \leq -2$. In words, x is greater than (or equal to) 1 *and* x is less than (or equal to) -2 . Now we must ask ourselves: What number is bigger than 1 and less than -2 ? *The Answer: There is no such number!*

Inequalities such as $1 \leq x \leq -2$ are often written by students who have an imperfect understanding of the concepts and notations they are using. I hope you are not one of these students. Exercise 1.5. ■

1.6. Since $w < 4$, we deduce that $w - 4 < 0$. From (7) we then have

$$\begin{aligned} |w - 4| &= -(w - 4) && \triangleleft \text{since } w - 4 < 0. \\ &= 4 - w. \end{aligned}$$

Exercise 1.6. ■

1.7. You should have used the same reasoning as in **EXAMPLE 1.3**. You must investigate the sign of $4t + 5$:

$$\begin{aligned}t < -3 &\implies 4t < -12 \\ &\implies 4t + 5 < -12 + 5 \\ &\implies 4t + 5 < -7\end{aligned}$$

Thus, $4t + 5 < -7 < 0$; that is, $4t + 5$ is a negative number. So

$$|4t + 5| = -(4t + 5) \quad \triangleleft \text{from (7)}$$

• **Question.** Suppose I had asked to you calculate $|4t + 15|$. What can be said about the sign of $4t + 15$ and the value of $|4t + 15|$?

Exercise 1.7. ■

1.8. Simply apply the **definition**:

$$|3x - 12| = \begin{cases} 3x - 12 & \text{if } 3x - 12 \geq 0 \\ -(3x - 12) & \text{if } 3x - 12 < 0 \end{cases}$$

This can and *should be* refined: $3x - 12 \geq 0$ is equivalent to $3x \geq 12$, which, in turn, is equivalent to $x \geq 4$. Thus,

$$|3x - 12| = \begin{cases} 3x - 12 & \text{if } x \geq 4 \\ -(3x - 12) & \text{if } x < 4 \end{cases}$$

One last refinement,

$$|3x - 12| = \begin{cases} 3x - 12 & \text{if } x \geq 4 \\ 12 - 3x & \text{if } x < 4 \end{cases}$$

That seems routine!

Exercise 1.8. ■

1.9. Simply apply the **definition**:

$$|2 + 3x| = \begin{cases} 2 + 3x & \text{if } 2 + 3x \geq 0 \\ -(2 + 3x) & \text{if } 2 + 3x < 0 \end{cases}$$

This can and *should be* refined: $2 + 3x \geq 0$ is equivalent to $3x \geq -2$, which, in turn, is equivalent to $x \geq -2/3$. Thus,

$$|2 + 3x| = \begin{cases} 2 + 3x & \text{if } x \geq -\frac{2}{3} \\ -(2 + 3x) & \text{if } x < -\frac{2}{3} \end{cases}$$

One last refinement,

$$|2 + 3x| = \begin{cases} 2 + 3x & \text{if } x \geq -\frac{2}{3} \\ -2 - 3x & \text{if } x < -\frac{2}{3} \end{cases}$$

Exercise 1.9. ■

1.10. *Solutions:*

(a) $|-56a| = 56|a|$

(b) $|(-1)^n x| = |(-1)^n| |x| = |-1|^n |x| = |x|.$

(c) $\left| \frac{x}{-10} \right| = \frac{|x|}{10}.$

Exercise 1.10. ■

1.11. The symbolic answer is $d(34, -5.3) = |34 - (-5.3)|$. This evaluates to

$$d(34, -5.3) = |34 - (-5.3)| = |34 + 5.3| = 39.3$$

Thus,

$$\boxed{d(34, -5.3) = 39.3}$$

Exercise 1.11. ■

1.12. The inequality $|x - 10| < 3$ states that x is less than 3 units away from the number 10.

Quiz. Where does that put x ? Between

- (a) 8 and 9 (b) 7 and 12 (c) 7 and 13 (d) 8 and 13

This reasoning is the beginning of solving absolute inequalities. More in **Lesson 8**. Exercise 1.12. ■

1.13. The expression $|x - (-4)|$, from equation (8), is interpreted as the distance between x and -4 ; or it is the distance x is away from -4 .

The inequality $|x - (-4)| < 3$ then states that x is less than 3 units away from the number -4 .

Quiz. Where does that put x ? Between

- (a) -7 and -4 (b) -6 and -1 (c) -4 and -1 (d) -7 and -1

Exercise 1.13. ■

1.14. The inequality $|x - 1| > 6$ can be interpreted as ' x is *more than* 6 units away from the number 1.' Exercise 1.14. ■

Solutions to Examples

1.1. We use a little algebraic notation. Let

$$N = 453.123123123123123\dots$$

Based on the pattern of the repeating decimal, multiply both sides of the equation by 1000

$$N = 453.123123123123123\dots \tag{S-1}$$

so

$$1000N = 453123.123123123123123\dots \tag{S-2}$$

Now subtract equation (S-1) from (S-2) to obtain,

$$999N = 452123 - 453 = 452670$$

thus,

$$N = \frac{452670}{999} = \frac{150890}{333}$$

(You did see that common factor of 3 didn't you?)

Solutions to Examples (continued)

Answer:

$$453.123123123123123 \dots = \frac{150890}{333}$$

Example 1.1. ■

1.2. We reason as follows.

Compute $|a - 5|$. To remove the absolute values, we *must* know the sign of $a - 5$. It is given that $a > 5$; this implies $a - 5 > 0$. Now since $a - 5$ is positive, we deduce from the definition of absolute value, equation (7), that

$$|a - 5| = a - 5 \quad \triangleleft \text{ since } a - 5 > 0$$

Now compute $|5 - a|$. To remove the absolute values, we *must* know the sign of $5 - a$. It is given that $a > 5$, and so $5 - a < 0$. Thus, from (7), we see that

$$|5 - a| = -(5 - a) = a - 5 \quad \triangleleft \text{ since } 5 - a < 0.$$

Example 1.2. ■

1.3. Here we are given $s > 6$. To compute the $|2s - 3|$ we need to know the sign of $2s - 3$. We reason as follows.

$$\begin{aligned} s > 6 &\implies 2s > 12 \\ &\implies 2s - 3 > 12 - 3 \\ &\implies 2s - 3 > 9 \end{aligned}$$

Thus, $2s - 3 > 9$, in particular, $2s - 3 > 0$. Thus,

$$|2s - 3| = 2s - 3 \quad \triangleleft \text{from (7)}$$

Example 1.3. ■

1.4. Simply apply the **definition**:

$$|4x - 2| = \begin{cases} 4x - 2 & \text{if } 4x - 2 \geq 0 \\ -(4x - 2) & \text{if } 4x - 2 < 0 \end{cases}$$

This can and *should be* refined: $4x - 2 \geq 0$ is equivalent to $4x \geq 2$, which, in turn, is equivalent to $x \geq 1/2$. Thus,

$$|4x - 2| = \begin{cases} 4x - 2 & \text{if } x \geq \frac{1}{2} \\ -(4x - 2) & \text{if } x < \frac{1}{2} \end{cases}$$

One last refinement,

$$|4x - 2| = \begin{cases} 4x - 2 & \text{if } x \geq \frac{1}{2} \\ 2 - 4x & \text{if } x < \frac{1}{2} \end{cases}$$

and we're out of here!

Example 1.4. ■

1.5. The inequality $|x - 3| < 1$ can be interpreted quite simply. The $|x - 3|$ is the distance x is away from the number 3. The statement $|x - 3| < 1$ says that the distance x is away from the number 3 is *less than* 1 unit. Or, rephrased, x is less than 1 unit away from the number 3.

Doesn't this mean that x is somewhere between 2 and 4?

Example 1.5. ■

Important Points

Important Points (continued)

That's Right! The symbol $\mathbb{N} \subseteq \mathbb{Z}$ means \mathbb{N} is a *subset* of \mathbb{Z} ; this means that every *natural number* is also an *integer*.

The symbol ' \subseteq ' means 'subset': We say $A \subseteq B$ if it is true that every element of A is also a member of B , or $p \in A$ implies $p \in B$.

Important Point ■

Important Points (continued)

What does $p \in \mathbb{Z}$ mean? The symbol $p \in \mathbb{Z}$ means

- “ p is a member of the set \mathbb{Z} ” or,
- “ p belongs to the set of **integers**” or, more simply,
- “ p is an integer.”

The symbol ‘ \in ’, which is the Greek letter epsilon, may be read as ‘is a member of’ or ‘belongs to’.

Important Point ■

Important Points (continued)

The product of two odd integers is an odd integer. You can see this by example (for example, $3 \times 5 = 15$). More interesting, can you prove it? Try to do so using the representations of an odd and even integer given in (3); a formal proof requires some *algebra*. :-)

Important Point ■

Important Points (continued)

What is wrong with the rational number $-\frac{8}{12}$? *Answer:* It's not reduced to its lowest terms! You did see that didn't you?

Of course, $-\frac{8}{12} = -\frac{2}{3}$. We don't want to present fractions that are not in reduced form!

Important Point ■

Important Points (continued)

The rational number system is a *closed number system* with respect to addition, subtraction, multiplication, and division. This means that

$$p, q \in \mathbb{Q}, \text{ implies } p + q, p - q, pq, \frac{p}{q} \in \mathbb{Q},$$

in the last case, we require $q \neq 0$.

Is \mathbb{N} closed with respect to addition, subtraction, multiplication, and division? The answer is **No**.

Question. With respect to which arithmetic operations is \mathbb{N} closed?

- (a) Addition, subtraction, multiplication
- (b) Addition, subtraction
- (c) Addition, multiplication
- (d) n.o.t.

If you did not get it right the first time, and after you have found the correct response, think things through again and try to understand the correctness of the answer.

Important Points (continued)

Question. With respect to which arithmetic operations is \mathbb{Z} closed?
(Choose the “best alternative.”)

- (a) Addition, subtraction, multiplication
- (b) Addition, subtraction
- (c) Addition, multiplication
- (d) n.o.t.

It is good to think about such things.

Important Point ■

Important Points (continued)

Well, in that case, read on!

Important Point ■

Important Points (continued)

The answer here is **No** in general. For example, the midpoint between 0 and 1 (integers) is $\frac{1}{2}$ (not an integer).

However, the midpoint between 0 and 2 (integers) is 1 (an integer).

Sometimes its true and sometimes its false. The reason the answer is **NO** is because the question was “Is it (always) true that the point midway between two integers is an integer?”

Important Point ■

Important Points (continued)

Repeat the solution to the exercise with modifications:

$$\begin{aligned}t < -3 &\implies 4t < -12 \\ &\implies 4t + 15 < -12 + 15 \\ &\implies 4t + 5 < 3\end{aligned}$$

We can say that $4t+5 < 3$ which means it may be positive or negative; therefore, we cannot simplify the $|4t + 15|$ in this case without more information as to the value of t .

Important Point ■